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An elliptic current operator for the eight-vertex model

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Abstract
We compute the operator which creates the missing degenerate states in the algebraic Bethe ansatz of the eight-vertex model at roots of unity and relate it to the concept of an elliptic current operator. We find that in sharp contrast with the corresponding formalism in the six-vertex model at roots of unity [12] the current operator is not nilpotent with the consequence that in the construction of degenerate eigenstates of the transfer matrix an arbitrary number of exact strings can be added to the set of regular Bethe roots. Thus the original set of free parameters $\{s, t\}$ of an eigenvector of the transfer matrix T is enlarged to become $\{s, t, \lambda_{c,1}, \ldots, \lambda_{c,n}\}$ with arbitrary string centres $\lambda_{c,j}$ and arbitrary n.

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1. Introduction

The spectrum of eigenvalues of the transfer matrix of the six-vertex model and of the Hamiltonian of the *XXZ* spin chain

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cos\pi\gamma\sigma_j^z \sigma_{j+1}^z\right\} \tag{1}$$

with periodic boundary conditions has many degenerate multiplets when γ is rational. Degenerate eigenvalues are always a signal of extra symmetries of the system and the degeneracies of the six-vertex model for rational γ have been extensively analysed [1–5] in terms of an sl_2 loop algebra symmetry.

The eight-vertex model of Baxter [6–11] is a lattice model whose transfer matrix is given by

$$\mathbf{T}_8(v)|_{\mu,\nu} = \operatorname{Tr} W(\mu_1, \nu_1) W(\mu_2, \nu_2) \cdots W(\mu_N, \nu_N), \tag{2}$$

where $\mu_j, \nu_j = \pm 1$ and $W(\mu, \nu)$ is a 2×2 matrix whose non-vanishing elements are given as

$$\begin{aligned} W(+1,+1)|_{+1,+1} = W(-1,-1)|_{-1,-1} &= \rho\Theta(2\eta)\Theta(\lambda-\eta)H(\lambda+\eta) = a(\lambda) \\ W(-1,-1)|_{+1,+1} = W(+1,+1)|_{-1,-1} &= \rho\Theta(2\eta)H(\lambda-\eta)\Theta(\lambda+\eta) = b(\lambda) \\ W(-1,+1)|_{+1,-1} = W(+1,-1)|_{-1,+1} &= \rho H(2\eta)\Theta(\lambda-\eta)\Theta(\lambda+\eta) = c(\lambda) \\ W(+1,-1)|_{+1,-1} = W(-1,+1)|_{-1,+1} &= \rho H(2\eta)H(\lambda-\eta)H(\lambda+\eta) = d(\lambda) \end{aligned} \tag{3}$$

where $H(u)$ and $\Theta(u)$ are Jacobi's theta functions defined in appendix A. In the following we set $\rho = 1$. This transfer matrix has a set of degeneracies similar to the six-vertex model when η is restricted to values η_0 given by

$$\eta_0 = 2m_1 K/L, \tag{4}$$

where L and m_1 are integers and K is the complete elliptic integral of the first kind. Some of these degeneracies were recognized in [7] but the full set seems to have only recently been obtained [12–14]. Corresponding to the six-vertex model these degeneracies also indicate the existence of a symmetry algebra over and beyond the star triangle equation [6, 10] which guarantees the commutation of the transfer matrix for different values of the spectral variable λ. This extra symmetry at rational values of η/K was first seen and exploited by Onsager [15] in his 1944 solution of the Ising model.

For the six-vertex model we have Chevalley generators [1], current operators [4], and the degenerate states can be characterized by a Drinfeld polynomial [4, 5]. For the eight-vertex model we know much less. Deguchi [16, 17] has produced a meromorphic function which should play the role of a Drinfeld polynomial but because expressions for neither the Chevalley nor the current generators are known it is not yet possible to make a connection of this meromorphic function with a symmetry algebra.

The purpose of this paper is to partially fill this gap by finding the operator which is the generalization of the current operator found for the six-vertex model [4] which we accordingly will call 'elliptic current operator'. We will do this in analogy with our previous computation [4] for the six-vertex model by using the algebraic Bethe ansatz of Takhtadzhan and Faddeev [18] for the eight-vertex model. This method is better adapted to the computation of the current operator than the related formalism of Felder and Varchenko [19, 20] used by Deguchi [16, 17].

In the description of degenerate eigenstates of the T matrix the concept of complete strings plays a central role. However there are two quite different meanings of this term and to avoid confusion we will summarize here several of the significant differences between the two concepts, the details of which will be fully elaborated in the text below. In a previous paper [12] we explained the dimensions of degenerate subspaces of T by using the Q-matrix constructed by Baxter in [6]. This Q-matrix is defined only when η satisfies the root of unity condition $2L\eta = 2m_1K + \mathrm{i}m_2K'$. To distinguish this Q matrix from Baxter's Q-matrix of [7, 10] which exists for generic η we denote it by Q_{72}. The eigenvalues of Q_{72} are quasiperiodic with $2\mathrm{i}K'$ as the imaginary quasiperiod. The eigenvalues of those eigenvectors of Q_{72} which are degenerate eigenstates of T have string-like subsets of zeros, which we will call Q-strings[3]. These Q-strings have fixed string centres which have the property that for every string centre λ_c there is another string centre at $\lambda_c + \mathrm{i}K'$. For each eigenstate the number of Q strings is fixed [12]. For more details see section 3, especially (23).

String-like sequences of spectral parameters (5) will also occur as arguments in products of B-operators building up an eigenvector of T. These eigenvectors are quasiperiodic with an imaginary quasiperiod $\mathrm{i}K'$. These strings will be called B-strings. They were first reported

[3] This definition rests on the properties of Q_{72} shown in equation (23).

in [7] and are further discussed in [11]. The centres of B-strings are free but because of the quasiperiodicity their imaginary parts are restricted to lie between zero and $\mathrm{i}K'$. Furthermore we will show below in section 5 that, in contrast with Q-strings, the number of B-strings used to produce an eigenstate is not fixed. These significant differences between the two types of strings mean that the two concepts are different and Q-strings cannot be considered as limiting or special cases of B strings.

In the following we compute the creation operator for B strings. We give this result in section 2 as (8)–(11). In section 3 we present results about the size of degenerate multiplets in the eight-vertex model at roots of unity. In section 4 we derive (8)–(11) and in section 5 we construct the operator for multiple B strings from the single B string operator (8). We conclude in section 6 with a discussion of our results.

2. Results

In order to present our result for the elliptic current operator we recall in appendix B the formalism and notations of the algebraic Bethe ansatz used in [18] to compute some eigenvectors in the root of unity case where (4) holds. Here we explain why there are states which are not given by this formalism and present our result for the operator which creates these missing states. The details of the computation of this creation operator are given in section 4.

Eigenvectors of the transfer matrix (2) are given by (B.25). They describe all non-degenerate eigenstates of T and because of their dependence on parameters s, t also some (but not all) degenerate eigenstates. There are, however, eigenvectors where the root of unity condition (4) holds for which the construction (B.25) is not adequate. This happens if we consider a set of λ_k given by

$$\lambda_k = \lambda_c - (k-1)2\eta_0, \qquad k = 1, \ldots, L_s, \tag{5}$$

where η_0 is given by (4) and λ_c is arbitrary. Such a set of λ_k is called a string and λ_c is called the string centre. The string length L_s is determined by the integer L occurring in equation (4). It will be shown that $L_s = L/2$ if L is even and $L_s = L$ if L is odd. The attempt to construct eigenstates of T containing B-strings by using (5) in (B.25) fails because the operator

$$B_l^{L_s}(\lambda_c) = B_{l+1,l-1}(\lambda_1)\cdots B_{l+L_s,l-L_s}(\lambda_{L_s}) \tag{6}$$

is found numerically (in agreement with the analogous analytic computation of Tarasov [21] for the six-vertex model) to vanish

$$B_l^{L_s}(\lambda_c) = 0 \tag{7}$$

for all l, λ_c, s, t.

Examples of states with a set of λ_k given by (5) have been known since the original work of [7–9, 12–14] and thus there must be a creation operator for these B-string states. This operator should be an elliptic generalization of the loop algebra current found for the six-vertex model in [4]. In the following sections we shall prove that the creation operator of complete B-strings is

$$B_l^{L_s,1}(\lambda_c) = \sum_{j=1}^{L_s} B_{l+1,l-1}(\lambda_1)\cdots\left(\frac{\partial B_{l+j,l-j}}{\partial\eta}(\lambda_j) - \hat{Z}_j\frac{\partial B_{l+j,l-j}}{\partial\lambda}(\lambda_j)\right)\cdots B_{l+L_s,l-L_s}(\lambda_{L_s}), \tag{8}$$

where

$$\hat{Z}_1(\lambda_c) = \frac{\hat{X}(\lambda_c)}{\hat{Y}(\lambda_c)} \tag{9}$$

with

$$\hat{X}(\lambda_c) = -2\sum_{k=0}^{L_s-1} k\frac{\omega^{-2(k+1)}\rho_{k+1}}{P_k P_{k+1}} \tag{10}$$

$$\hat{Y}(\lambda_c) = \sum_{k=0}^{L_s-1} \frac{\omega^{-2(k+1)}\rho_{k+1}}{P_k P_{k+1}} \tag{11}$$

and

$$\hat{Z}_j(\lambda_c) = \hat{Z}_1(\lambda_c - (j-1)2\eta_0), \tag{12}$$

where $\omega = e^{2\pi i m/L}$ is a Lth root of unity

$$\rho_k = h^N(\lambda_c - (2k-1)\eta_0) \tag{13}$$

and

$$P_k = \prod_{m=1}^{n_r} h(\lambda_c - \lambda_m^r - 2k\eta_0). \tag{14}$$

In (14) λ_m^r (which we call regular Bethe roots) satisfy the Bethe equation (B.26).

The functions $\hat{X}(\lambda_c)$ and $\hat{Y}(\lambda_c)$ generalize to the eight-vertex model the functions $X(\lambda_c)$ and $Y(\lambda_c)$ previously obtained [4] for the six-vertex model. The function $\hat{Y}(\lambda_c)$ has been previously obtained by Deguchi [16, 17].

3. Rank of degenerate subspaces

We first present new exact results [12] about the size of degenerate multiplets of eigenvalues of the transfer matrix of the eight-vertex model. It is known that in the six-vertex model at roots of unity the degenerate subspaces of eigenstates of the transfer matrix are related to evaluation representations of the sl_2 loop algebra. If the zeros of the Drinfeld polynomial of a highest weight state have multiplicity 1 the dimension of the multiplet is a power of 2. This is the generic case. It is remarkable that the dimension of degenerate subspaces of the set of eigenstates of the transfer matrix of the eight-vertex model for rational η/K is also a power of 2. This has been shown in [12] with quite a different method. Because of its importance in the present context we give a short description of the result in a more complete form.

In [6] Baxter constructed a Q-matrix for the eight-vertex model under the restriction that

$$2L\eta = 2m_1 K + i m_2 K'. \tag{15}$$

We denote this Q-matrix here by Q_{72} to distinguish it from the Q-matrix which Baxter defined in [7] and [10]. We note that there occurs a slightly different restriction on η in the construction of eigenstates of the eight-vertex transfer matrix [7–9, 18]

$$L\eta = 2m_1 K + i m_2 K'. \tag{16}$$

In this work we consider only the case $m_2 = 0$.

In (15) and (16) L, m_1, m_2 are integers. We will have to use both restrictions depending on the context. We have shown in [12] that Q_{72} *does not exist if in (15) L is odd and m_1 is even*. Q_{72} has the properties [12]

$$Q_{72}(v + 2K) = (-1)^{\nu'} Q_{72}(v) \tag{17}$$

$$Q_{72}(v + 2iK') = q^{-N}\exp(-iN\pi v/K)Q_{72}(v), \tag{18}$$

where ν' is the eigenvalue of the operator

$$S = \sigma_3 \otimes \cdots \otimes \sigma_3 \tag{19}$$

which commutes with the transfer matrix T. From this we derived the most general form for the eigenvalues of Q_{72} [12]:

$$Q_{72}(v) = \mathcal{K}(q; v_k) \exp(-\mathrm{i}\nu\pi v/2K) \prod_{j=1}^{N} H(v - v_j). \tag{20}$$

The integers ν and ν' satisfy

$$\nu + \nu' + N = \text{even integer} \tag{21}$$

$$N + \left(-\nu i K' + \sum_{j=1}^{N} v_j\right) \Big/ K = \text{even integer} \tag{22}$$

which follows from equations (17) and (18). If subsets of roots form exact strings, the explicit form of (20) is

$$Q_{72}(v) = \hat{\mathcal{K}}(q; v_k) \exp(\mathrm{i}(n_B - \nu)\pi v/2K) \prod_{j=1}^{n_B} h\big(v - v_j^B\big)$$

$$\times \prod_{j=1}^{n_L} H(v - \mathrm{i}w_j) H(v - \mathrm{i}w_j - 2K/L) \cdots H(v - \mathrm{i}w_j - 2(L-1)K/L) \tag{23}$$

$$2n_B + Ln_L = N. \tag{24}$$

n_B is the number of Bethe roots v_k and n_L the number of exact Q-strings of length L. *Note that n_L is always even.* The eigenvalues of the transfer matrix are (equation (C.38) of [6]):

$$t(v)Q_{72}(v) = h^N(v - \eta)Q_{72}(v + 2\eta) + h^N(v + \eta)Q_{72}(v - 2\eta) \tag{25}$$

or

$$t(v) = \exp\left(\frac{\mathrm{i}\pi(n_B - \nu)\eta}{K}\right) h^N(v - \eta) \prod_{j=1}^{n_B} \frac{h\big(v - v_j^B + 2\eta\big)}{h\big(v - v_j^B\big)}$$

$$+ \exp\left(-\frac{\mathrm{i}\pi(n_B - \nu)\eta}{K}\right) h^N(v + \eta) \prod_{j=1}^{n_B} \frac{h\big(v - v_j^B - 2\eta\big)}{h\big(v - v_j^B\big)}, \tag{26}$$

where according to equations (22) and (23)

$$\nu = n_B + \left(2\sum_{j=1}^{n_B} \mathrm{Im}\, v_j^B + L\sum_{j=1}^{n_L} w_j\right) \Big/ K'. \tag{27}$$

It follows from a functional equation satisfied by the eigenvalues of Q_{72} (see [12] equations (3.11) and (3.12)) that for any given set of Bethe roots v_k^B there are 2^{n_L} independent solutions w_l. Then

$$w_j = w_j^0 + \epsilon_j K' \qquad w_j^0 < K', \qquad \epsilon_j = 0, 1, \qquad j = 1, \ldots, n_L \tag{28}$$

and

$$\nu = n_B + \nu_0 + L\sum_{j=1}^{n_L} \epsilon_j \qquad \text{with integer} \qquad \nu_0 = \left(2\sum_{j=1}^{n_B} \mathrm{Im}\, v_j^B + L\sum_{j=1}^{n_L} w_j^0\right) \Big/ K' \tag{29}$$

$$t(v) = \exp\left(-\mathrm{i}\pi m \sum_{j=1}^{n_L} \epsilon_j\right) \hat{t}(v) \tag{30}$$

$$\hat{t}(v) = \exp(-\mathrm{i}\pi \nu_0 \eta/K) h^N(v-\eta) \prod_{j=1}^{n_B} \frac{h\left(v - v_j^B + 2\eta\right)}{h\left(v - v_j^B\right)} + \exp(\mathrm{i}\pi \nu_0)\eta/K) h^N(v+\eta) \prod_{j=1}^{n_B} \frac{h\left(v - v_j^B - 2\eta\right)}{h\left(v - v_j^B\right)}. \tag{31}$$

As m is an odd integer we find that for a fixed set of Bethe roots v_j^B, $j = 1, \ldots, n_B$ the 2^{n_L} independent solutions w_l give 2^{n_L-1} eigenstates of T with eigenvalue $\hat{t}(v)$ and 2^{n_L-1} eigenstates of T with eigenvalue $-\hat{t}(v)$. Q_{72} does not exist for even m but we get a result also for even m and odd L using

$$t(v+K; K-\eta) = (-1)^{\nu'} t(v;\eta) = \exp\left(-\mathrm{i}\pi(\nu' + m) \sum_{j=1}^{n_L} \epsilon_j\right) \hat{t}(v). \tag{32}$$

It follows from equations (21) and (29) that

$$n_B + \nu_0 + L \sum_{j=1}^{n_L} \epsilon_j + \nu' = 2r \tag{33}$$

$$\nu' + m \sum_{j=1}^{n_L} \epsilon_j = 2r - n_B - \nu_0 + (m-L) \sum_{j=1}^{n_L} \epsilon_j. \tag{34}$$

As m and L are both odd the right-hand side is either even or odd for *all* 2^{n_L} independent solutions w_l. Consequently the degenerate multiplet has 2^{n_L} elements. This delivers much more detailed information on the size of degenerate multiplets than what was previously known. The only precise information is found in [7–9] where $2N$ special eigenvectors of T are constructed with degenerate subsets of size $2N/L$. The main purpose of this paper is to describe a method to construct the complete set of eigenstates of the transfer matrix corresponding to these 2^{n_L} sets of exact Q-strings. It is well known that the eigenvectors of the transfer matrix of the eight-vertex model depend on parameters s, t as shown in [9, 18]. It shall be demonstrated in the following that this freedom allows the construction of a small subset of degenerate states but is insufficient to generate the full degenerate subspaces.

4. The root of unity limit

4.1. Exact B-strings in the eight-vertex model

We have shown that the length of complete Q-strings in the set of zeros of eigenvalues of Q_{72} at $\eta = mK/L$ is L. The results of Baxter [7–9] and Takhtadzhan and Faddeev [18] for the eight-vertex model are obtained for $\eta = 2m_1K/L$ where m_1, L are integers which include also those η for which Q_{72} does not exist. We now determine the length of complete B-strings in this formalism. Baxter was the first to discuss the existence of complete B-strings in the Bethe ansatz solution of the eight-vertex model at roots of unity $L\eta = 2m_1K$ (page 54 in [9]). He defines them as strings of Bethe roots $u_j = u_c + 2j\eta$ which will cancel out of the TQ-equation and Bethe's equations. We apply this idea here to equation (B.26). The contribution P_s of a string of length L_s

$$\lambda_k = \lambda_c - (k-1)2\eta_0, \qquad k = 1, \ldots, L_s \tag{35}$$

to the right-hand side of

$$\frac{h^N(\lambda_j+\eta)}{h^N(\lambda_j-\eta)} = \exp(-4\pi \mathrm{i} m/L)\frac{\prod_{k=1,k\neq j}^{n} h(\lambda_j-\lambda_k+2\eta_0)}{\prod_{k=1,k\neq j}^{n} h(\lambda_j-\lambda_k-2\eta_0)} \tag{36}$$

is after cancellations

$$P_s = \frac{h(\lambda_j-\lambda_c+(L_s-1)2\eta_0)h(\lambda_j-\lambda_c+L_s2\eta_0)}{h(\lambda_j-\lambda_c-2\eta_0)h(\lambda_j-\lambda_c)}. \tag{37}$$

We find that

$$P_s = 1 \qquad \text{if} \quad 2L_s\eta_0 = r2K, \tag{38}$$

where r is an integer

$$2L_s m_1 = rL. \tag{39}$$

As $(m_1, L) = 1$ it follows $r = r_1 m_1$ and $2L_s = r_1 L$.

For *even* L we get the smallest possible B-string length $L_s = L/2$ by setting $r_1 = 1$

Then $\eta = 2m_1 K/L = m_1 K/L_s$ where m_1 is odd. We note that this is the case in which Q_{72} exists.

For *odd* L we get the smallest possible B-string length $L_s = L$ by setting $r_1 = 2$. Then $\eta = 2m_1 K/L_s$ and Q_{72} does not exist.

The resulting string length is

$$L_s = L/2 \qquad \text{if} \quad L = \text{even}, \qquad L_s = L \qquad \text{if} \quad L = \text{odd}. \tag{40}$$

In the rest of this subsection we show that exact B-strings change the eigenvalue of the transfer matrix by a factor ± 1 in agreement with the results (30)–(34). The eigenvalue of the transfer matrix is given in terms of Bethe roots

$$\tilde{\Lambda} = \exp(2\pi \mathrm{i} m/L)h^N(\lambda+\eta)\frac{\prod_{j=1,}^{n} h(\lambda-\lambda_j-2\eta_0)}{\prod_{j=1}^{n} h(\lambda-\lambda_j)} + \exp(-2\pi \mathrm{i} m/L)h^N(\lambda-\eta)\frac{\prod_{j=1}^{n} h(\lambda-\lambda_j+2\eta_0)}{\prod_{j=1}^{n} h(\lambda-\lambda_j)}. \tag{41}$$

A B-string of length L_s contributes to the first product in equation (25)

$$P_1 = \frac{h(\lambda-\lambda_c-2\eta_0)}{h(\lambda-\lambda_c-(L_s-1)2\eta_0)} \tag{42}$$

and to the second product

$$P_2 = \frac{h(\lambda-\lambda_c-L_s 2\eta_0)}{h(\lambda-\lambda_c)}. \tag{43}$$

We find that $P_1 = 1$ and $P_2 = 1$ if $L_s 2\eta_0 = 4K \times$ integer and $P_1 = -1$ and $P_2 = -1$ if $L_s 2\eta_0 = 2K \times$ odd integer.

The result is

(a) A single B-string changes the sign of the eigenvalue of T if $L =$ even.
(b) For L =odd the sign of the eigenvalue is unchanged.

This result is consistent with corresponding results obtained from the properties of Q_{72} in the preceding subsection. We recall that the number Q-strings in the set of eigenvalues of Q_{72} is always even. In that case the sign of the eigenvalue of $t(v)$ is determined by the factor $\exp(\mathrm{i}(n_B - \nu)\pi v/2K)$ in (23). Q_{72} exists for $\eta =$ (odd integer)$/L_s$ or for $m_1 = 2 \times$ (odd integer), $L = 2L_s$. In that case both signs of $t(v)$ occur. This corresponds to case (a). When $\eta =$ (even integer)/(odd L_s) we get from (32) that there is no sign change like in case (b).

The results are summarized in table 1.

Table 1. The string size and properties of Q_{72} for $L\eta = 2m_1 K$, $(m_1, L) = 1$. The number of L-strings n_L is defined in (24).

L	m_1	L_s	η	Q_{72}	Size of deg. subspace
Odd	Even or odd	L	Even int. $\times$ K/odd int.	Does not exist	2^{n_L}
Even	Odd	$L/2$	Odd int. $\times$ K/integer	Exists	2^{n_L-1}

4.2. Construction of B-string creation operators

We construct the operator $B_l^{L,1}$ for $\eta = \eta_0 = 2m_1 K/L$ by using the formalism of the algebraic Bethe ansatz [18] by writing

$$\eta = \eta_0 + \epsilon \tag{44}$$

and letting $\epsilon \to 0$. Thus we consider the operator

$$\chi_l = B_l^L(\lambda_c) \prod_{m=L_s+1}^{n} B_{l+m,l-m}(\lambda_m) \tag{45}$$

and the vector

$$\psi_l = \chi_l \Omega_N^{l-n}, \tag{46}$$

where λ_j, $j = L_s + 1, \ldots, n$ are at this stage arbitrary and where

$$B_l^L(\lambda_c) = B_{l+1,l-1}(\lambda_1) \cdots B_{l+L_s,l-L_s}(\lambda_{L_s}) \tag{47}$$

with arguments

$$\lambda_k = \lambda_c - 2(k-1)\eta_0 - \hat{Z}_k(\lambda_c)\epsilon, \qquad k = 1, \ldots, L_s \tag{48}$$

will give the desired creation operator when $\epsilon \to 0$. The B-string length is given in equation (40). The allowed number of B operators in (45) will be determined later. The key ingredient of our method is the function $\hat{Z}_k(\lambda_c)$ in equation (48). We do not obtain a result by simply setting $\lambda_k = \lambda_c - 2(k-1)(\eta_0 + \epsilon)$. The function $\hat{Z}_k(\lambda_c)$ has to be chosen such that $\sum_l \omega^l \psi_l$ (with appropriately defined sum) becomes an eigenvector of the transfer matrix. The fundamental relations which allow the construction of eigenstates of the transfer matrix of the eight-vertex model are derived using the commutations relations (B.14)–(B.17)

$$\begin{aligned} A_{l,l}(\lambda)\chi_l(\lambda_1, \ldots, \lambda_n) &= \kappa(\lambda, \lambda_1, \ldots, \lambda_n)\chi_{l-1}(\lambda_1, \ldots, \lambda_n)A_{l+n,l-n}(\lambda) \\ &\quad + \sum_{j=1}^{n} \kappa_j^l(\lambda, \lambda_1, \ldots, \lambda_n)\chi_{l-1}(\lambda_1, \ldots, \lambda_{j-1}, \lambda, \lambda_{j+1}, \ldots, \lambda_n)A_{l+n,l-n}(\lambda_j) \end{aligned} \tag{49}$$

$$\begin{aligned} D_{l,l}(\lambda)\chi_l(\lambda_1, \ldots, \lambda_n) &= \tilde{\kappa}(\lambda, \lambda_1, \ldots, \lambda_n)\chi_{l+1}(\lambda_1, \ldots, \lambda_n)D_{l+n,l-n}(\lambda) \\ &\quad + \sum_{j=1}^{n} \tilde{\kappa}_j^l(\lambda, \lambda_1, \ldots, \lambda_n)\chi_{l+1}(\lambda_1, \ldots, \lambda_{j-1}, \lambda, \lambda_{j+1}, \ldots, \lambda_n)D_{l+n,l-n}(\lambda_j). \end{aligned} \tag{50}$$

The coefficients $\kappa, \tilde{\kappa}, \kappa_k^l, \tilde{\kappa}_k^l$ are

$$\kappa(\lambda, \lambda_1, \ldots, \lambda_n) = \prod_{k=1}^{n} \alpha(\lambda, \lambda_k), \qquad \tilde{\kappa}(\lambda, \lambda_1, \ldots, \lambda_n) = \prod_{k=1}^{n} \alpha(\lambda_k, \lambda) \tag{51}$$

$$\kappa_j^l(\lambda, \lambda_1, \ldots, \lambda_n) = -\beta_{l-1}(\lambda, \lambda_j) \prod_{k=1, k\neq j}^{n} \alpha(\lambda_j, \lambda_k) \tag{52}$$

$$\tilde{\kappa}_j^l(\lambda, \lambda_1, \ldots, \lambda_n) = \beta_{l+1}(\lambda, \lambda_j) \prod_{k=1, k\neq j}^{n} \alpha(\lambda_k, \lambda_j). \tag{53}$$

We must carefully expand (46) and (51)–(53) to first order as $\epsilon \to 0$.

The expansion of $B_{k,l}(\lambda)$ is

$$B_{l+m,l-m}(s, t, \lambda_m, \eta) = B_{l+m,l-m}(s, t, \lambda_c, \eta_0) + \epsilon \left(\frac{\partial B_{l+m,l-m}}{\partial \eta}(s, t, \lambda_c, \eta_0) - \hat{Z}_m \frac{\partial B_{l+m,l-m}}{\partial \lambda}(s, t, \lambda_c, \eta_0) \right) \tag{54}$$

if λ_m is of type (48).

4.3. The expansion of κ and κ_k^l

The coefficients $\kappa(\lambda, \lambda_1, \ldots, \lambda_n)$ and $\tilde{\kappa}(\lambda, \lambda_1, \ldots, \lambda_n)$ are in the limit $\epsilon \to 0$ given by

$$\begin{aligned} \kappa(\lambda, \lambda_1, \ldots, \lambda_n) &= (-1)^{L+1} \kappa^r(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) \\ \tilde{\kappa}(\lambda, \lambda_1, \ldots, \lambda_n) &= (-1)^{L+1} \tilde{\kappa}^r(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) \end{aligned} \tag{55}$$

$$\kappa^r(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) = \prod_{k=L_s+1}^{n} \alpha(\lambda, \lambda_k) \qquad \tilde{\kappa^r}(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) = \prod_{k=L_s+1}^{n} \alpha(\lambda_k, \lambda) \tag{56}$$

as the B-string part becomes ± 1 because of periodicity. This guarantees that $B_l^{L,1}$ if applied to an eigenstate does change the eigenvalue at most by changing its sign. The superscript r indicates that κ^r and $\tilde{\kappa^r}$ depend only on those arguments λ_j, $j = L_s + 1, \ldots, n$ which finally will be set to be regular Bethe roots.

We further observe that on the right-hand side of (52) all factors are of order ϵ^0 except the factor $\alpha(\lambda_k, \lambda_{k+1}) = O(\epsilon^1)$ with $1 \leqslant k \leqslant L_s$ and equivalently on the right-hand side of (53) all factors are of order ϵ^0 except $\alpha(\lambda_{k-1}, \lambda_k) = O(\epsilon^1)$ with $1 \leqslant k \leqslant L_s$. It follows for $1 \leqslant k \leqslant L_s$ that

$$\kappa_k^l(\lambda) = \epsilon(-1)^{L+1}(\hat{Z}_{k+1} - \hat{Z}_k - 2) \frac{h'(0)h(\tau_{l-1} + \lambda_{0k} - \lambda)}{h(\tau_{l-1})h(\lambda_{0k} - \lambda)} \frac{\prod_{m=L_s+1}^{n} h(\lambda_{0k} - \lambda_m - 2\eta_0)}{\prod_{m=L_s+1}^{n} h(\lambda_{0k} - \lambda_m)} \tag{57}$$

$$\tilde{\kappa}_k^l(\lambda) = -(-1)^{L+1}\epsilon(\hat{Z}_k - \hat{Z}_{k-1} - 2) \frac{h'(0)h(\tau_{l+1} + \lambda_{0k} - \lambda)}{h(\tau_{l+1})h(\lambda_{0k} - \lambda)} \frac{\prod_{m=L_s+1}^{n} h(\lambda_{0k} - \lambda_m + 2\eta_0)}{\prod_{m=L_s+1}^{n} h(\lambda_{0k} - \lambda_m)}. \tag{58}$$

The variables

$$\lambda_{0k} = \lambda_c - 2(k-1)\eta_0, \qquad k = 1, \ldots, L_s \tag{59}$$

are the B-string arguments and the variables λ_m, $m = L_s + 1, \ldots, n$ are still arbitrary and will be chosen finally as regular roots. If however $k > L_s$ the factor related to the B-string is $(-1)^{L+1}$ and the limit is of order ϵ^0

$$\begin{aligned} \kappa_k^l(\lambda, \lambda_1, \ldots, \lambda_n) &= (-1)^{L+1} \kappa_k^{r,l}(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) \\ \tilde{\kappa}_k^l(\lambda, \lambda_1, \ldots, \lambda_n) &= (-1)^{L+1} \tilde{\kappa}_k^{r,l}(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) \end{aligned} \tag{60}$$

$$\kappa_k^{r,l}(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) = -\beta_{l-1}(\lambda, \lambda_k) \prod_{j=L_s+1, j\neq k}^{n} \alpha(\lambda_k, \lambda_j)$$
$$\tilde{\kappa}_k^{r,l}(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) = \beta_{l+1}(\lambda, \lambda_k) \prod_{j=L_s+1, j\neq k}^{n} \alpha(\lambda_j, \lambda_k). \tag{61}$$

4.4. Expansion of state vectors

We may now expand the vector ψ_l by using the expansion (54) of $B_{k,l}$ in (46) to find that $\psi_l = \psi_l^{(0)} + \epsilon \psi_l^{(1)}$ where

$$\psi_l^{(1)} = B_l^{L,1}(\lambda_c) \prod_{m=L_s+1}^{n} B_{l+m,l-m}(\lambda_m) \Omega_N^{l-n} \tag{62}$$

and

$$B_l^{L,1}(\lambda_c) = \sum_{j=1}^{L_s} B_{l+1,l-1}(\lambda_1) \cdots \left(\frac{\partial B_{l+j,l-j}}{\partial \eta}(\lambda_j) - \hat{Z}_j \frac{\partial B_{l+j,l-j}}{\partial \lambda}(\lambda_j) \right) \cdots B_{l+L_s,l-L_s}(\lambda_{L_s}). \tag{63}$$

The allowed number of operators B in (62) follows from the periodicity properties of $A_{k,l}, \ldots, D_{k,l}$ for $\eta = \eta_0 = 2m_1 K/L$. Their period in k, l is L. It follows that

$$A_{l+n,l-n}(\lambda)\Omega_N^{l-n} = A_{l+n+r_1 L, l-n-r_2 L}(\lambda)\Omega_N^{l-n} = h^N(\lambda+\eta)\Omega_N^{l-1-n} \tag{64}$$

and

$$D_{l+n,l-n}(\lambda)\Omega_N^{l-n} = D_{l+n+r_1 L, l-n-r_2 L}(\lambda)\Omega_N^{l-n} = h^N(\lambda-\eta)\Omega_N^{l+1-n} \tag{65}$$

if

$$2n + (r_1 + r_2)L = N \tag{66}$$

for integers r_1, r_2 When we insert (55) and (57) into (49) we find

$$\begin{aligned} A_{l,l}(\lambda)\psi_l^{(1)} = {} & (-1)^{L+1}\Lambda(\lambda)\psi_{l-1}^{(1)} + (-1)^{L+1}\frac{h'(0)}{h(\tau_{l-1})}\sum_{k=1}^{L_s}(\hat{Z}_{k+1} - \hat{Z}_k - 2)\frac{h(\tau_{l-1} + \lambda_{0k} - \lambda)}{h(\lambda_{0k} - \lambda)} \\ & \times h^N(\lambda_{0k} + \eta_0)\frac{\prod_{m=L_s+1}^{n} h(\lambda_{0k} - \lambda_m - 2\eta_0)}{\prod_{m=L_s+1}^{n} h(\lambda_{0k} - \lambda_m)} \\ & \times \psi_{l-1}(\lambda_{01}, \ldots, \lambda_{0,k-1}, \lambda, \lambda_{0,k+1}, \ldots, \lambda_{0L_s}, \ldots, \lambda_n) \\ & + \sum_{k=L_s+1}^{n} \Lambda_k^l \psi_{l-1}^{(1)}(\lambda_{01}, \ldots, \lambda_{0L_s}, \ldots, \lambda_{k-1}, \lambda, \lambda_{k+1}, \ldots, \lambda_n) \end{aligned} \tag{67}$$

and when we insert (55) and (58) into (50) we obtain

$$\begin{aligned} D_{l,l}(\lambda)\psi_l^{(1)} = {} & (-1)^{L+1}\tilde{\Lambda}(\lambda)\psi_{l+1}^{(1)} - (-1)^{L+1}\frac{h'(0)}{h(\tau_{l+1})}\sum_{k=1}^{L_s}(\hat{Z}_k - \hat{Z}_{k-1} - 2)\frac{h(\tau_{l+1} + \lambda_{0k} - \lambda)}{h(\lambda_{0k} - \lambda)} \\ & \times h^N(\lambda_{0k} - \eta_0)\frac{\prod_{m=L_s+1}^{n} h(\lambda_{0k} - \lambda_m + 2\eta_0)}{\prod_{m=L_s+1}^{n} h(\lambda_{0k} - \lambda_m)} \\ & \times \psi_{l+1}(\lambda_{01}, \ldots, \lambda_{0,k-1}, \lambda, \lambda_{0,k+1}, \ldots, \lambda_{0L_s}, \ldots, \lambda_n) \\ & + \sum_{k=L_s+1}^{n} \tilde{\Lambda}_k^l \psi_{l+1}^{(1)}(\lambda_{01}, \ldots, \lambda_{0L_s}, \ldots, \lambda_{k-1}, \lambda, \lambda_{k+1}, \ldots, \lambda_n), \end{aligned} \tag{68}$$

where

$$\Lambda(\lambda) = h^N(\lambda+\eta)\kappa^r(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) \qquad \tilde{\Lambda}(\lambda) = h^N(\lambda-\eta)\tilde{\kappa}^r(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) \tag{69}$$

$$\Lambda_k^l(\lambda) = (-1)^{L+1} h^N(\lambda+\eta)\kappa_k^{r,l}(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n) \tag{70}$$

$$\tilde{\Lambda}_k^l(\lambda) = (-1)^{L+1} h^N(\lambda-\eta)\tilde{\kappa}_k^{r,l}(\lambda, \lambda_{L_s+1}, \ldots, \lambda_n). \tag{71}$$

We remark that on the right-hand sides of (67) and (68) in the first sum $\psi_{l-1}(\lambda_{01}, \ldots, \lambda_{0,k-1}, \lambda, \lambda_{0,k+1}, \ldots, \lambda_n)$ is $O(\epsilon^0)$ because there is not a complete B-string: λ_{0k} is replaced by λ. In the last line however where $k > L_s$ the B-string is complete and therefore the first-order term $\psi_{l-1}^{(1)}$ has to be taken.

We add (67) and (68), multiply by $\omega^l = \exp(2\pi i m l/L)$ and sum over l. After the shift $l \to l+1$ in the first sum and $l \to l-1$ in the second sum we get

$$\begin{aligned}
T\sum_{l=0}^{L-1}\omega^l\psi_l^{(1)} &= \sum_{l=0}^{L-1}(A_{l,l}(\lambda)+D_{l,l}(\lambda))\omega^l\psi_l^{(1)} \\
&= (-1)^{L+1}(\omega\Lambda+\omega^{-1}\tilde{\Lambda})\sum_{l=0}^{L-1}\omega^l\psi_l^{(1)} + (-1)^{L+1}h'(0)\sum_{l=0}^{L-1}\omega^l\sum_{k=1}^{L_s}\frac{h(\tau_l+\lambda_{0k}-\lambda)}{h(\tau_l)h(\lambda_{0k}-\lambda)} \\
&\left[\omega(\hat{Z}_{k+1}-\hat{Z}_k-2)h^N(\lambda_{0k}+\eta_0)\frac{\prod_{m=L_s+1}^n h(\lambda_{0k}-\lambda_m-2\eta_0)}{\prod_{m=L_s+1}^n h(\lambda_{0k}-\lambda_m)}\right. \\
&\qquad \left. -\,\omega^{-1}(\hat{Z}_k-\hat{Z}_{k-1}-2)h^N(\lambda_{0k}-\eta_0)\frac{\prod_{m=L_s+1}^n h(\lambda_{0k}-\lambda_m+2\eta_0)}{\prod_{m=L_s+1}^n h(\lambda_{0k}-\lambda_m)}\right] \\
&\psi_l(\lambda_{01}, \ldots, \lambda_{0,k-1}, \lambda, \lambda_{0,k+1}, \ldots, \lambda_{L_s}, \ldots, \lambda_n) \\
&\qquad + \sum_{l=0}^{L-1}\omega^l\sum_{k>L_s}\left(\omega\Lambda_k^{l+1}+\omega^{-1}\tilde{\Lambda}_k^{l-1}\right)\psi_l^{(1)}(\lambda_{01}, \ldots, \lambda_{0L_s}, \ldots, \lambda_{k-1}, \lambda, \lambda_{k+1}, \ldots, \lambda_n)
\end{aligned} \tag{72}$$

4.5. Determination of $\hat{Z}_k$

In order that $\sum_{l=0}^{L-1}\omega^l\psi_l$ be an eigenvector all terms on the right-hand side of (72) must vanish except the first. The arguments of [18] show that the last term will vanish if the $\lambda_k, k = L_s+1, \ldots, n$ are chosen to satisfy Bethe's equation (B.26). The remaining unwanted terms in equation (72) will vanish if we set

$$\begin{aligned}
\omega(\hat{Z}_{k+1}-\hat{Z}_k-2)h^N(\lambda_c-(2k-3)\eta_0)\prod_{m=L_s+1}^n h(\lambda_c-\lambda_m-2k\eta_0) \\
= \omega^{-1}(\hat{Z}_k-\hat{Z}_{k-1}-2)h^N(\lambda_c-(2k-1)\eta_0)\prod_{m=L_s+1}^n h(\lambda_c-\lambda_m-2(k-2)\eta_0)
\end{aligned} \tag{73}$$

To simplify the notation we recall the definitions of ρ_k and P_k in (13) and (14) and further define

$$f_k = \hat{Z}_{k+1} - \hat{Z}_k - 2, \tag{74}$$

where

$$\hat{Z}_k = \hat{Z}_1(\lambda - 2(k-1)\eta_0). \tag{75}$$

Thus (73) is rewritten as

$$\omega f_k \rho_{k-1} P_k = \omega^{-1} f_{k-1} \rho_k P_{k-2}. \tag{76}$$

We define $\hat{f}_k = \omega^{2k} f_k$ and get

$$\hat{f}_k \rho_{k-1} P_k = \hat{f}_{k-1} \rho_k P_{k-2} \tag{77}$$

$$\frac{\hat{f}_k}{\hat{f}_{k-1}} = \frac{\rho_k P_{k-2}}{\rho_{k-1} P_k} = \frac{\rho_k P_{k-2} P_{k-1}}{\rho_{k-1} P_{k-1} P_k}. \tag{78}$$

It follows that

$$\hat{f}_k = \tilde{g} \frac{\rho_k}{P_{k-1} P_k} \tag{79}$$

and

$$f_k = \tilde{g} \frac{\omega^{-2k} \rho_k}{P_{k-1} P_k} \tag{80}$$

$\tilde{g}$ follows from $\sum_k \hat{f}_k = -2L_s$:

$$\tilde{g} = -\frac{2L_s}{\sum_{k=0}^{L_s-1} \frac{\omega^{-2k}\rho_k}{P_{k-1}P_k}} = -\frac{2L_s}{\sum_{k=0}^{L_s-1} \frac{\omega^{-2(k+1)}\rho_{k+1}}{P_k P_{k+1}}}. \tag{81}$$

As $Q_k = \frac{\rho_k}{P_{k-1} P_k}$ satisfies $Q_{k+L_s} = Q_k$ we find that

$$\tilde{g}(\lambda + 2\eta_0) = \omega^2 \tilde{g}(\lambda). \tag{82}$$

It follows from equation (80) that

$$\hat{Z}_1 - \hat{Z}_0 - 2 = \tilde{g} \frac{\rho_0}{P_{-1} P_0}. \tag{83}$$

We make the ansatz

$$\hat{Z}_1(\lambda) = \tilde{g} \sum_{k=0}^{L_s-1} c_k \frac{\rho_{k+1}}{P_k P_{k+1}}. \tag{84}$$

It follows

$$\hat{Z}_0(\lambda) = \hat{Z}_1(\lambda + 2\eta_0) = \tilde{g}(\lambda + 2\eta_0) \sum_{k=0}^{L_s-1} c_{k+1} \frac{\rho_{k+1}}{P_k P_{k+1}} = \omega^2 \tilde{g}(\lambda) \sum_{k=0}^{L_s-1} c_{k+1} \frac{\rho_{k+1}}{P_k P_{k+1}} \tag{85}$$

and

$$\hat{Z}_1 - \hat{Z}_0 - 2 = \tilde{g} \sum_{k=0}^{L_s-1} \left(c_k - \omega^2 c_{k+1} + \frac{1}{L_s} \omega^{-2(k+1)} \right) \frac{\rho_{k+1}}{P_k P_{k+1}}. \tag{86}$$

Comparison of (83) with (86) gives

$$c_k - \omega^2 c_{k+1} + \frac{1}{L_s} \omega^{-2(k+1)} = 0, k = 0, \ldots, L_s - 2 \tag{87}$$

and

$$c_{L_s-1} - \omega^2 c_{L_s} = 1 - \frac{1}{L_s}. \tag{88}$$

We define $c_k = \omega^{-2(k+1)} \hat{c}_k$. Equations (87) and (88) are then simplified to

$$\hat{c}_k - \hat{c}_{k+1} = -\frac{1}{L_s}, \qquad k = 0, \ldots, L_s - 2 \tag{89}$$

and

$$\hat{c}_{L_s-1} - \hat{c}_{L_s} = 1 - \frac{1}{L_s}. \tag{90}$$

This gives the coefficients

$$c_k = \omega^{-2(k+1)} \frac{k}{L_s} \qquad k = 0, \ldots, L_s - 1, \quad c_{L_s} = c_0 \tag{91}$$

$$\hat{Z}_1 = -2 \frac{\sum_{k=0}^{L_s-1} k \frac{\omega^{-2(k+1)} \rho_{k+1}}{P_k P_{k+1}}}{\sum_{k=0}^{L_s-1} \frac{\omega^{-2(k+1)} \rho_{k+1}}{P_k P_{k+1}}}. \tag{92}$$

Thus we have obtained the desired result (8) for $B_l^{L,1}$.

5. Multiple *B*-strings

We use a more compact notation to show the essentials of the proof that the result (92) and (8) is correct also for multiple B-strings. Let ψ_2 denote a state with two substrings like (6).

$$\psi_2 = B^L(\lambda_{c_1}) B^L(\lambda_{c_2}) B^{\text{reg}}. \tag{93}$$

We first show that equation (49) is then of order ϵ^2. It is sufficient to show this for

$$\sum_{k=1}^{n} \Lambda_k^l(\lambda, \lambda_1, \ldots \lambda_n) \psi_{l-1}(\lambda_1, \ldots, \lambda_{k-1}, \lambda, \lambda_{k+1}, \ldots, \lambda_n). \tag{94}$$

We write

$$\Lambda_k^l(\lambda, \lambda_1, \ldots, \lambda_n) = -\beta_{l-1}(\lambda, \lambda_k) h^N(\lambda + \eta) \prod_{j=1, j\neq k}^{n} \alpha(\lambda_k, \lambda_j) = p_1 p_2 p_{\text{reg}}$$

$$p_1 = \prod_{j=1, j\neq k}^{L_s} \alpha(\lambda_k, \lambda_j) \qquad p_2 = \prod_{j=L_s+1, j\neq k}^{2L_s} \alpha(\lambda_k, \lambda_j) \qquad p_{\text{reg}} = \prod_{j=2L_s+1, j\neq k}^{n} \alpha(\lambda_k, \lambda_j).$$

Each term of the sum in (94) has the form

$$\Lambda_k^l(\lambda, \lambda_1, \ldots, \lambda_n) \psi_{l-1}(\lambda_1, \ldots, \lambda_{k-1}, \lambda, \lambda_{k+1}, \ldots, \lambda_n) \sim p_1 p_2 p_{\text{reg}} B^L(\lambda_{c_1}) B^L(\lambda_{c_2}) B^{\text{reg}}.$$

In the limit $\epsilon \to 0$ the order of the power of ε depends on the position of k.

	p_1	p_2	p_{reg}	$B^L(\lambda_{c_1})$	$B^L(\lambda_{c_2})$	B^{reg}
$k \in S_1$	(ϵ^1)	(ϵ^0)	(ϵ^0)	(ϵ^0)	(ϵ^1)	(ϵ^0)
$k \in S_2$	(ϵ^0)	(ϵ^1)	(ϵ^0)	(ϵ^1)	(ϵ^0)	(ϵ^0)
$k \in R_0$	(ϵ^0)	(ϵ^0)	(ϵ^0)	(ϵ^1)	(ϵ^1)	(ϵ^0),

where $S_1 = \{1, 2, \ldots, L_s\}$, $S_2 = \{L_s + 1, L_s + 2, \ldots, 2L_s\}$, $R_0 = \{2L_s + 1, 2L_s + 2, \ldots, n\}$.

It is evident that equation (49) is then of second order. It also follows that terms of the first and second LINES of the preceding list are removed by (92) and terms from the last line add up to zero if the variables λ_k for $k \in R_0$ are regular roots.

The total number of operators B building an eigenvector is restricted by

$$2(n_B + n_s) + rL = N \tag{95}$$

(see equation (66)) where n_B is the number of regular roots and n_s is the number of roots belonging to B-strings. It is important to note that the integer r may be positive and negative. It follows that there is no restriction on the number of B-string-operators in a state vector.

To elucidate the role of B-strings we note that the analytical expression for eigenstates of the eight-vertex model depends on two free parameters s, t. For degenerate eigenstates which form a space of dimension d this means that a subspace of dimension $d_0 < d$ can be constructed by the variation of s and t without applying the elliptic current operator. Detailed numerical studies have revealed that the variation of s, t will only give the full degenerate eigenspace for very small d (e.g. $d = 2$). In all other cases one needs to use the elliptic current operator with the additional freedom to choose the string centre to generate the full subspace. After this is achieved by adding a certain number of B-strings (the exact number depends on the system size N and the value of η) the addition of more and more B-strings will only map this subspace into itself. In particular, adding B-strings to a singlet state with $n_B = N/2$ Bethe roots does not destroy this state but reproduces it. It is not this redundancy which is important, but the prospect that this might have to do with a cyclic nature of the hidden symmetry. Finally we stress that the addition of B-strings does not single out a special basis as the string centres can be chosen randomly. The numerical tests have been performed for spin chains of length $N = 6, 8, 10, 12$ and crossing parameters $\eta = K/2, K/3, 2K/3$.

6. Discussion of the result

In this paper we have constructed the creation operator of B-strings in the algebraic Bethe ansatz of the eight-vertex model. We have made no connection with a symmetry algebra and therefore we are unable to even define what is meant by a current operator. Nevertheless we have used the phrase 'elliptic current operator' in the title because the operator $B_l^{L,1}(\lambda_c)$ is obviously a generalization of the current operator $B_6^{(L)}(v)$ of the six-vertex model at roots of unity [4]. The term 'generalization' is used here admittedly in a vague sense. The true relation of the operator (8) to the six-vertex current operator derived in [4] can only be found by a careful analytical investigation of the trigonometric limit of (8). The six-vertex limit of eigenvectors of the eight-vertex model with B-string number 0 is presented in great detail in [22] which will presumably be of great help also in understanding the limit of the B-string operator (8).

The operator $B_l^{L,1}$ serves as generator of missing eigenstates in the eigenspaces of degenerate eigenvalues of T. The eigenstates (B.25) of the algebraic Bethe ansatz depend on parameters s, t. However, in the case that an eigenvalue is degenerate one obtains only a subspace of the related degenerate eigenspace by varying s, t in (B.25). The complete set of missing states is found by adding a sufficient number of B-strings. This is similar to the string operators $B_6^{(L)}(v)$ found in [4] which generate the missing eigenstates in the algebraic Bethe ansatz of the six-vertex model at roots of unity.

Tools necessary to construct all eigenstates of T in the coordinate Bethe ansatz of the six- and eight-vertex models at roots of unity have been developed by Baxter [11]. There is however a marked difference between our and Baxter's result. In the coordinate Bethe ansatz [11] the total number of roots (the sum of the number of regular roots and the number of roots which are elements of exact strings) is $\leqslant N$. (See page 50 in [9]) whereas our method allows the addition of an arbitrary number of B-strings even though the rank of the generated eigenspace is the same in both cases. The six-vertex model at roots of unity has a loop sl_2 symmetry algebra [1] and in (1.37) of [4] we produced the operator $B_6^{(L)}(v)$ which creates L strings for the six-vertex model at roots of unity and argued that this is the current operator for the loop sl_2 symmetry algebra. This operator has poles at the zeros of the Laurent polynomial

$$Y(v) = \sum_{l=0}^{L-1} \frac{\sinh^N \frac{1}{2}(v - (2l+1)\mathrm{i}\gamma_0)}{\prod_{k=1}^{n} \sinh \frac{1}{2}(v - v_k - 2\mathrm{i}l\gamma_0) \prod_{k=1}^{n} \sinh \frac{1}{2}(v - v_k - 2\mathrm{i}(l+1)\gamma_0)} \qquad (96)$$

in $\exp(Lv)$ which we identified with the Drinfeld polynomial. The operator $B_6^{(L)}(v)$ is the generator of the elements of the algebra in the mode basis of the loop algebra. By differentiation of $E^-(z)$ (see (1.19) in [4]) with respect to z^{-1} and then setting $z^{-1} = 0$ one can extract the mode operators of the irreducible representation related to a Bethe state. These conjectures have recently been given representation theory foundation by Deguchi [5]. It follows from the representation theory of the loop sl_2 algebra that when the roots of the Drinfeld polynomial are distinct that the space of states generated by $B_6^{(L)}(v)$ is a highest weight finite dimensional representation which is the direct sum of spin 1/2 representations. Accordingly the current operator $B_6^{(L)}(v)$ will be nilpotent.

The operator $B_l^{L,1}(\lambda_c)$ in (8) is, in form, the eight-vertex generalization of the operator $B_6^{(L)}(v)$ in equation (1.37) of [4] and the meromorphic function $\hat{Y}(\lambda_c)$ in (11) (which is essentially the function $G(z)$ in (5.8) of [17] and (31) of [16]) appears to be the generalization of the Drinfeld polynomial $Y(v)$ (96) of the six-vertex model. We therefore expect that the elliptic B-string operator $B_l^{(L,1)}(\lambda_c)$ is the current generator of the elliptic symmetry at roots of unity.

There is, however, an important difference between $B_6^{(L)}(v)$ and $B_l^{L,1}(\lambda_c)$. We saw in the previous section that there is no restriction on the number of B-strings in a state. Therefore, in contrast to $B_6^{(L)}(v)$ the operator $B_l^{L,1}(\lambda_c)$ is not nilpotent and therefore the space of states produced by the action of $B_l^{L,1}(\lambda_c)$ is not a highest weight representation as was the case in the six-vertex model. Here again an investigation of the six-vertex limit like in [22] will give us valuable insight into the nature of symmetry.

It remains to relate the concept of the B-strings of this paper with the Q-strings of [12] and to compute the degeneracy of the states from the zeros of the meromorphic function $\hat{Y}(\lambda_c)$ (11). For the six-vertex model it followed [4] from the representation theory of sl_2 loop algebra that the number of degenerate states is 2 raised to the power of the order of the Drinfeld polynomial (96) if the zeros of the Drinfeld polynomial are distinct. For the Q-strings in the eight-vertex model the degeneracy followed [12] from the fact that for every Q-string centre λ_c there was another independent Q-string centre at $\lambda_c + \mathrm{i}K'$. These two concepts need to be united in a representation theory of the elliptic algebra which underlies the eight-vertex model.

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Appendix A. Theta functions

The definition of Jacobi theta functions of nome q is

$$H(v) = 2\sum_{n=1}^{\infty}(-1)^{n-1}q^{(n-\frac{1}{2})^2}\sin[(2n-1)\pi v/(2K)] \tag{A.1}$$

$$\Theta(v) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(nv\pi/K), \tag{A.2}$$

where K and K' are the standard elliptic integrals of the first kind and

$$q = \mathrm{e}^{-\pi K'/K}. \tag{A.3}$$

These theta functions satisfy the quasi periodicity relations (15.2.3) of [10]

$$H(v+2K) = -H(v) \tag{A.4}$$
$$H(v+2\mathrm{i}K') = -q^{-1}\,\mathrm{e}^{-\pi \mathrm{i} v/K} H(v) \tag{A.5}$$

and

$$\Theta(v+2K) = \Theta(v) \tag{A.6}$$
$$\Theta(v+2\mathrm{i}K') = -q^{-1}\,\mathrm{e}^{-\pi \mathrm{i} v/K}\Theta(v). \tag{A.7}$$

$\Theta(v)$ and $H(v)$ are not independent but satisfy (15.2.4) of [10]

$$\begin{aligned} \Theta(v+\mathrm{i}K') &= \mathrm{i}q^{-1/4}\,\mathrm{e}^{-\frac{\pi \mathrm{i} v}{2K}} H(v) \\ H(v+\mathrm{i}K') &= \mathrm{i}q^{-1/4}\,\mathrm{e}^{-\frac{\pi \mathrm{i} v}{2K}} \Theta(v). \end{aligned} \tag{A.8}$$

Appendix B. The algebraic Bethe ansatz

We follow the formalism of [18] and define what is called the monodromy matrix by

$$\mathcal{T} = \mathcal{L}_N \cdots \mathcal{L}_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{B.1}$$

where $\mathcal{L}_n$ is a 2×2 matrix in auxiliary space with entries which are 2×2 matrices in spin space acting on the nth spin in the spin chain and A, B, C, D are $2^N \times 2^N$ matrices in spin space.

$$\mathcal{L}_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \tag{B.2}$$

$$\alpha_n = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad \beta_n = \begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix} \qquad \gamma_n = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}, \qquad \delta_n = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \tag{B.3}$$

a, b, c and d are defined in equation (3).

A gauge transformed monodromy matrix is then defined as

$$\mathcal{T}_{k,l} = M_k^{-1}(\lambda)\mathcal{T}(\lambda)M_l(\lambda) = \begin{pmatrix} A_{k,l} & B_{k,l} \\ C_{k,l} & D_{k,l} \end{pmatrix}, \tag{B.4}$$

where the matrices M_k introduced by Baxter [8] are

$$M_k = \begin{pmatrix} x_k^1 & y_k^1 \\ x_k^2 & y_k^2 \end{pmatrix} \tag{B.5}$$

$$\begin{pmatrix} x_k^1(\lambda) \\ x_k^2(\lambda) \end{pmatrix} = \begin{pmatrix} H(s+2k\eta-\lambda) \\ \Theta(s+2k\eta-\lambda) \end{pmatrix} \qquad \begin{pmatrix} y_k^1(\lambda) \\ y_k^2(\lambda) \end{pmatrix} = \frac{1}{g(\tau_k)} \begin{pmatrix} H(t+2k\eta+\lambda) \\ \Theta(t+2k\eta+\lambda) \end{pmatrix} \tag{B.6}$$

$$g(u) = H(u)\Theta(u), \qquad \tau_l = (s+t)/2 + 2l\eta - K \tag{B.7}$$

with

$$\det M_k = \frac{2g(\lambda+(t-s)/2)}{g(K)} \equiv m(\lambda). \tag{B.8}$$

Thus (B.4) is explicitly written as

$$A_{k,l}(\lambda) = \frac{1}{m(\lambda)}\big(y_k^2(\lambda)x_l^1(\lambda)A(\lambda) + y_k^2(\lambda)x_l^2(\lambda)B(\lambda) - y_k^1(\lambda)x_l^1(\lambda)C(\lambda) - y_k^1(\lambda)x_l^2(\lambda)D(\lambda)\big) \tag{B.9}$$

$$B_{k,l}(\lambda) = \frac{1}{m(\lambda)}\big(y_k^2(\lambda)y_l^1(\lambda)A(\lambda) + y_k^2(\lambda)y_l^2(\lambda)B(\lambda) - y_k^1(\lambda)y_l^1(\lambda)C(\lambda) - y_k^1(\lambda)y_l^2(\lambda)D(\lambda)\big) \tag{B.10}$$

$$C_{k,l}(\lambda) = \frac{1}{m(\lambda)}\big(-x_k^2(\lambda)x_l^1(\lambda)A(\lambda) - x_k^2(\lambda)x_l^2(\lambda)B(\lambda) + x_k^1(\lambda)x_l^1(\lambda)C(\lambda) + x_k^1(\lambda)x_l^2(\lambda)D(\lambda)\big) \tag{B.11}$$

$$D_{k,l}(\lambda) = \frac{1}{m(\lambda)}\big(-x_k^2(\lambda)y_l^1(\lambda)A(\lambda) - x_k^2(\lambda)y_l^2(\lambda)B(\lambda) + x_k^1(\lambda)y_l^1(\lambda)C(\lambda) + x_k^1(\lambda)y_l^2(\lambda)D(\lambda)\big) \tag{B.12}$$

and in this notation the transfer matrix (2) becomes

$$T(\lambda) = A(\lambda) + D(\lambda) = A_{l,l}(\lambda) + D_{l,l}(\lambda), \tag{B.13}$$

where we note that the sum $A_{l,l}(\lambda) + D_{l,l}(\lambda)$ is independent of l even though each term in the sum separately depends on l.

Using relations which Baxter derived in [8] the following commutation relations are obtained [18]:

$$B_{k,l+1}(\lambda)B_{k+1,l}(\mu) = B_{k,l+1}(\mu)B_{k+1,l}(\lambda) \tag{B.14}$$

$$A_{k,l}(\lambda)B_{k+1,l-1}(\mu) = \alpha(\lambda,\mu)B_{k,l-2}(\mu)A_{k+1,l-1}(\lambda) - \beta_{l-1}(\lambda,\mu)B_{k,l-2}(\lambda)A_{k+1,l-1}(\mu) \tag{B.15}$$

$$D_{k,l}(\lambda)B_{k+1,l-1}(\mu) = \alpha(\mu,\lambda)B_{k+2,l}(\mu)D_{k+1,l-1}(\lambda) + \beta_{k+1}(\lambda,\mu)B_{k+2,l}(\lambda)D_{k+1,l-1}(\mu), \tag{B.16}$$

where

$$\alpha(\lambda,\mu) = \frac{h(\lambda-\mu-2\eta)}{h(\lambda-\mu)}, \qquad \text{and} \qquad \beta_k(\lambda,\mu) = \frac{h(2\eta)h(\tau_k+\mu-\lambda)}{h(\tau_k)h(\mu-\lambda)}, \tag{B.17}$$

and where

$$h(u) = \Theta(0)\Theta(u)H(u). \tag{B.18}$$

The commutation relations (B.14)–(B.16) are valid for all values of η.

In order to proceed further a set of direct product vectors is defined by

$$\Omega_N^l = \omega_1^l \otimes \cdots \otimes \omega_N^l \tag{B.19}$$

$$\omega_n^l = \begin{pmatrix} H(s+2(n+l)\eta-\eta) \\ \Theta(s+2(n+l)\eta-\eta) \end{pmatrix} \tag{B.20}$$

and it is shown [18] that

$$C_{N+l,l}\Omega_N^l = 0 \tag{B.21}$$

$$A_{N+l,l}\Omega_N^l = h^N(\lambda+\eta)\Omega_N^{l-1} \tag{B.22}$$

$$D_{N+l,l}\Omega_N^l = h^N(\lambda-\eta)\Omega_N^{l+1}. \tag{B.23}$$

Finally define the set of vectors

$$\psi_l(\lambda_1,\ldots,\lambda_n) = B_{l+1,l-1}(\lambda_1)\cdots B_{l+n,l-n}(\lambda_n)\Omega_N^{l-n}. \tag{B.24}$$

Then for the case that η satisfies the root of unity condition (4) and N is even one obtains the eigenstates

$$\Psi_m = \sum_{l=0}^{L-1} e^{2\pi iml/L} \psi_l(\lambda_1, \ldots, \lambda_n), \tag{B.25}$$

where $\lambda_1, \ldots, \lambda_n$ are chosen to satisfy what are called Bethe's equations

$$\frac{h^N(\lambda_j + \eta)}{h^N(\lambda_j - \eta)} = e^{-4\pi im/L} \prod_{k=1,k\neq j}^{n} \frac{h(\lambda_j - \lambda_k + 2\eta)}{h(\lambda_j - \lambda_k - 2\eta)}. \tag{B.26}$$

with $N = 2n + \text{integer} \times L$ and $m = 0, 1, \ldots, L - 1$.

It is important to note that all equations are valid for generic values of η/K except for the final expression (B.25) for the eigenvectors which is the only place where the root of unity condition (4) is used.

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